2.2 – Evaluating Determinants by Row Reduction

Theorem 2.2.1 Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

Theorem 2.2.2 Let A be a square matrix. Then $\det(A) = \det\left(A^T\right)$.

Theorem 2.2.3 Elementary Row Operations
Let A be an $n \times n$ matrix.
a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then $\det(B) = k\det(A)$.
b) If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.
c) If B is the matrix that results when a multiple of one row of A is added to another or when a multiple of one column is added to another, then $\det(B) = \det(A)$.

pf (a): Let B be the matrix that results from multiplying the i^{th} row of A by k.

$\det(A) = \sum_{l=1}^{n} a_{il} C_{il}$ by expanding along the i^{th} row

$$\det(B) = \sum_{l=1}^{n} b_{il} C_{il} = \sum_{l=1}^{n} k a_{il} C_{il} = k \sum_{l=1}^{n} a_{il} C_{il}$$

$$= k \det(A)$$

Consider $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 6 & 8 \\ 5 & 7 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 5 & 7 & 2 \end{vmatrix}$

Note: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$ but $2\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 2(ad-bc)$

Theorem 2.2.4 Let E be an $n \times n$ elementary matrix.
a) If E results from multiplying a single row of I_n by a nonzero number k, then $\det(E) = k$.
b) If E results from interchanging two rows of I_n, then $\det(E) = -1$.
c) If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.

This is a special case of Thm 2.2.3

#12 Evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion.

$$\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$$

$R_3 \to R_3 +$ ____

~~$R_3 \to 2R_3 +$~~

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 17/2 \end{vmatrix} = -17$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 5R_1$

$R_3 \to R_3 + \frac{13}{2}R_2$

B

$$= (-2)\left(\frac{17}{2}\right) \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{vmatrix} = -17$$

A

R_2 of B is R_2 of A times -2

$\Rightarrow \det(B) = -2\det(A)$

In #16 and 20, evaluate the determinant, given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6.$

#16 $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$ $R_1 \leftrightarrow R_3$

$= -(-6) = 6$

#20 $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix}$ $R_2 \to 2R_2$

$R_3 \to R_3 + 3R_1$

$= -12$

Theorem 2.2.5 If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

A row operation would yield a row of zeros. $\to$ see Thm 2.2.1.

Pf of 2.2.3 (b), (c)

(a) Mathematical Induction

Base case: 2×2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $\det(A) = ad - bc$. Let $A' = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

Then $\det(A') = cb - ad = -\det(A)$ ✓

Assume $n > 2$ and that the row-interchange property holds for matrices smaller than $n \times n$.

Let A be an $n \times n$ matrix and B be the matrix resulting from interchanging rows i and j of A. Expanding along
(or columns)
row k $(k \neq i, k \neq j)$ uses cofactors $(-1)^{k+r} |A_{kr}|$ and $(-1)^{k+r} |B_{kr}|$, where $|A_{kr}|$ and $|B_{kr}|$ are determinants.

Since these are of size $(n-1) \times (n-1)$, the cofactors have opposite signs.

Cofactor expansion yields $\det(A) = -\det(B)$.

(c) Let $\vec{a}_i = [a_{i1} \; a_{i2} \cdots a_{in}]$ be the i^{th} row of A. Let B be the same as A except that the j^{th} row $(j \neq i)$ results from adding $k\vec{a}_i$ to $\vec{a}_j$, the j^{th} row of A.

Then $\vec{b}_j = [ka_{i1} + a_{j1} \quad ka_{i2} + a_{j2} \; \cdots \; ka_{in} + a_{jn}]$.

Since $\vec{a_i} = \vec{b_i}$ for $i \neq j$, the minors $|A_{jr}|$ and $|B_{jr}|$ are equal for each r.

$$\det(B) = \sum_{r=1}^{n} b_{jr} C_{jr} = \sum_{r=1}^{n} (k a_{ir} + a_{jr}) C_{jr}$$

$$= \sum_{r=1}^{n} k a_{ir} C_{jr} + \sum_{r=1} a_{jr} C_{jr}$$

$$= k \det(D) + \det(A),$$

where D is the matrix obtained by replacing the j^{th} row of A with the i^{th} row of A.

Thus the i^{th} row of D and j^{th} row of D are equal. By part (b), we can interchange rows i and j of D to show

$$\det(D) = -\det(D) \Rightarrow \det(D) = 0$$

$$\Rightarrow \det(B) = \det(A).$$